Introduction to Integral Equation Methods

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Outline

• Introduction
• Mathematical preliminary
• Numerical methods
  • Discretization
  • Numerical quadratures
  • Fast algorithms
• Numerical examples
Setting

Solving linear BVPs:

\[ Lu = 0, \text{ in } \Omega \]
\[ u = g, \text{ on } \Gamma := \partial\Omega, \]

- \( L \) is a 2\textsuperscript{nd} order diff. op. (elliptic/parabolic).
- \( \Omega \in \mathbb{R}^d \) (\( d = 2, 3 \)) can be interior or exterior (decay conditions at \( \infty \) for the exterior case).
- Other boundary conditions: \( \partial u / \partial n = g \), or mix.
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- Other boundary conditions: \( \partial u / \partial n = g \), or mix.
- For simplicity, consider \( \Delta u = 0 \).
Mathematical preliminary: potential theory
The function

\[ G(x, y) = \begin{cases} \frac{1}{2\pi} \ln \frac{1}{|x-y|}, & d = 2 \\ \frac{1}{4\pi} \frac{1}{|x-y|}, & d = 3 \end{cases} \]

is called the fundamental solution of Laplace’s equation. For fixed \( y \in \mathbb{R}^d \) it is harmonic in \( \mathbb{R}^d \setminus \{y\} \).
Definition (fundamental solution)

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Theorem (Green's identity)

Let \( \Omega \) be a bounded domain of class \( C^1 \), and let \( u \in C^2(\Omega) \cap C^1(\overline{\Omega}) \) satisfy \( \Delta u = 0 \). Then \( u \) satisfies

\[ u(x) = \int_{\Gamma} G(x, y) \frac{\partial u(y)}{\partial n} \, ds_y - \int_{\Gamma} \frac{\partial}{\partial n_y} G(x, y) u(y) \, ds_y, \]

where \( \Gamma = \partial \Omega \) is the boundary of \( \Omega \), and \( x \in \Omega \).
Definition (layer potentials)

The integral operators

\[ S[\sigma] := \int_{\Gamma} G(x, y) \sigma(y) \, ds_y \]

\[ D[\mu] := \int_{\Gamma} \frac{\partial}{\partial n_y} G(x, y) \mu(y) \, ds_y \]

are called a single layer potential and a double layer potential respectively.
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Green’s identity:

\[
u(x) = S[\frac{\partial u}{n}] - D[u]. \quad x \in \Omega
\]
Potential theory

Theorem (the jump relation)

Let $\Gamma$ be of class $C^2$ and $\sigma \in C(\Gamma)$. Then the single layer potential $u = S[\sigma]$ is continuous throughout $\mathbb{R}^d$. It satisfies $\Delta u = 0$ for $x \notin \Gamma$. On the boundary there holds:

\[
\lim_{x \to x_0 \in \Gamma} \frac{\partial u(x)}{\partial n} = \frac{1}{2} \sigma(x_0) + \frac{\partial}{\partial n} S[\sigma](x_0)
\]

\[
\lim_{x \to x_0 \in \Gamma} \frac{\partial u(x)}{\partial n} = -\frac{1}{2} \sigma(x_0) + \frac{\partial}{\partial n} S[\sigma](x_0)
\]
Theorem (the jump relation)

Let $\Gamma$ be of class $C^2$ and $\mu \in C(\Gamma)$. Then the double layer potential $v = D[\mu]$ satisfies $\Delta v = 0$ for $x \notin \Gamma$, and can be continuously extended to the boundary from the interior or the exterior, with limiting values

$$
\lim_{\substack{x \to x_0 \\ x \in \Omega}} v(x) = -\frac{1}{2} \mu(x_0) + D[\mu](x_0)
$$

$$
\lim_{\substack{x \to x_0 \\ x \notin \Omega}} v(x) = \frac{1}{2} \mu(x_0) + D[\mu](x_0).
$$
Boundary integral equation (BIE)

Turning back to the Green’s identity

\[ u(x) = S[\frac{\partial u}{\partial n}] - D[u] \quad (\text{if } \Delta u = 0 \text{ in } \Omega), \]

letting \( x \to \Gamma \), we obtain (e.g. for the interior Dirichlet problem)

\[ S[\sigma] = \frac{1}{2} g(x) + D[g]. \]

- An integral equation (first kind Fredholm) for the unknown density function \( \sigma \).
- Once \( \sigma \) is obtained, the solution can be recovered by

\[ u(x) = S[\sigma] - D[g]. \]

- Unknowns on the boundary only.
Boundary integral equation (BIE)

Alternatively, we seek solution of the form $u(x) = D[\mu]$, where $\mu$ is an unkown density function supported on the boundary. $u(x)$ satisfies $\Delta u = 0$ automatically for $x \in \Omega$. It only remains to enforce the boundary condition. Letting $x \to \Gamma$, we obtain (e.g. for the interior Dirichlet problem)

$$-\frac{1}{2} \mu + D[\mu] = g.$$

- An integral equation (second kind Fredholm) for the unkown density function $\mu$, well conditioned. $\text{Cond}(\mathcal{P}) = ||\mathcal{P}|| \cdot ||\mathcal{P}^{-1}||$.
- Once $\mu$ is obtained, the solution can be recovered by
  $$u(x) = D[\mu].$$
- Unkowns on the boundary only.
Numerical methods
Discretization of the BIE

Consider the BIE

\[(I + K)\sigma = f,\]

where \(K\sigma = \int_{\Gamma} K(x, y)\sigma ds_y\), \(K\) is weakly singular when \(x = y\).

Task I: representation of \(\Gamma\)

- global (simple smooth geoms)
- local (adaptive and/or CAD geoms)

peri. trap. rule

G-L panels
Discretization of the BIE

Three types of discretization

- **Nyström:** replace the integral by quadrature and impose the BIE at nodes (points $\rightarrow$ points)
  \[ \sigma_i + \sum_j w_j K(x_i, x_j)\sigma_j = f(x_i) \]

- **Galerkin:** project into basis: \( \sigma = \sum_{m=1}^{N} \alpha_m \varphi_m \) (basis $\rightarrow$ basis)
  \[ \sum_{n=1}^{N} [(\varphi_m, \varphi_n) + (\varphi_m, K\varphi_n)]\alpha_n = (\varphi_m, f) \]

- **Collocation:** project \( \sigma \) into basis, impose the BIE at nodes (basis $\rightarrow$ points).
  \[ \sum_{n=1}^{N} (\varphi_n(x_m) + K\varphi_n(x_m))\alpha_n = f(x_m). \]

At the same order, accuracy basically the same. Nyström is the simplest to implement, and is naturally compatible with fast algorithms.
Quadrature tasks and challenges

Consider Nyström: \( \sigma_i + (\mathcal{K}\sigma)(x_i) = f_i \).

- Begin with a smooth ("native") quadrature rule

\[
\int_{\Gamma} g(x) ds_x = \sum_{j=1}^{N} w_j g(x_j). \quad g \in C^\infty(\Gamma)
\]

- Challenge 1: filling in the matrix. i.e. approx \( \mathcal{K}\sigma(x_i) \) on surface \( \Gamma \). weak singularity of \( K(x, y) \) when \( x = y \).

- Challenge 2: recovering the solution. i.e. approx \( \mathcal{K}\sigma(x) \) off surface \( \Gamma \).

Exponential in \( N \), but rate depends on target \( x \) (Barnett '14)
Quadrature: ideas

- on-surface (Review in 2D: (Hao–Barnett–Martinsson–Young '14)) auxiliary nodes, (analytic) complex Cauchy integrals, etc.

- off-surface: QBX, hedgehog, etc.
Fast algorithms

Discretization of $(I + \mathcal{K})\sigma = f$ leads to a dense $N \times N$ linear system.

- Fast multipole methods (FMM) reduces the cost of applying $\mathcal{K}\sigma$ to $O(N)$.
  - (Greengard-Rokhlin '87; Greengard-Rokhlin '97; Cheng-Greengard-Rokhlin '99)
  - 3D FMM lib developed at Flatiron: https://fmm3d.readthedocs.io
  - Combine with GMRES. Well conditioning $\Rightarrow O(1)$ iterations.

- Fast direct solvers.
  - Talk by Manas Rachh.
  - Suitable for: low rank perturbation of $\mathcal{K}$, multiple RHS, MFS, etc.
Numerical examples

Example 1: Inhomogeneous heat equation on a unit box with periodic boundary condition: automatic adaptivity
Numerical examples

Example 2: suspensions of rigid ellipses in shearing viscous flow
Summary

Integral equation methods have become powerful tools for the numerical solution of PDEs. They have the remarkable benefits:

- High accuracy
- Optimal/Near optimal complexity
- Ability to handle complex geometry
- Compatibility with automatic adaptivity

Requires a lot of machinery:

- Well conditioned integral formulation
- Efficient numerical quadratures
- Fast algorithms