Hierarchical low rank compression, with an application in quantum many-body physics

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CCM/CCQ

October 19, 2020

On work with Denis Golež of CCQ

Matrix structures based on low rank decomposition

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  - *Low rank* and *hierarchical low rank* structure

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- Yes – TSVD gives $\mathcal{O}(N)$ algorithm

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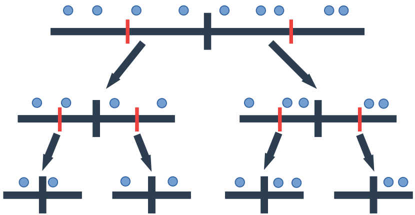

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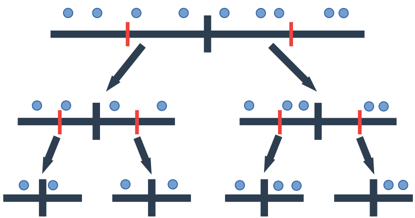
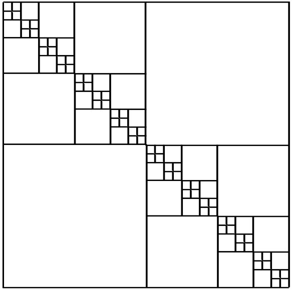
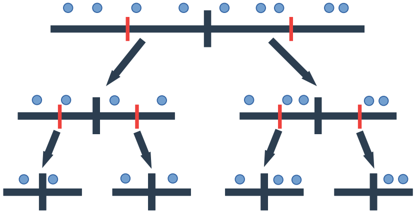
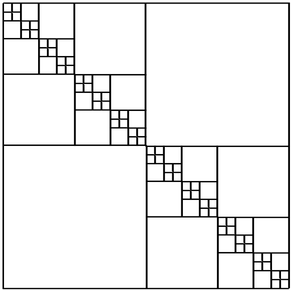




Hierarchical off-diagonal low rank (HODLR) matrices





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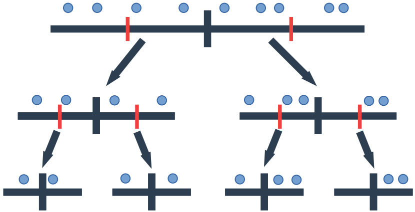

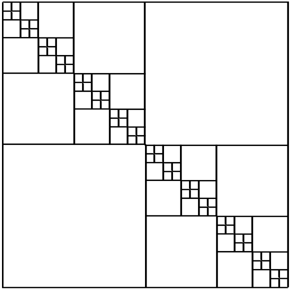

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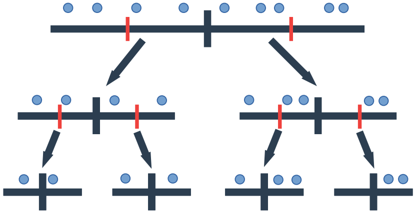

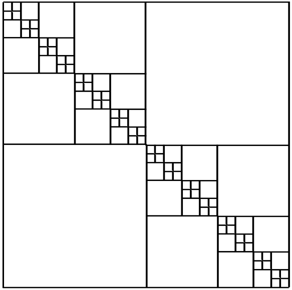

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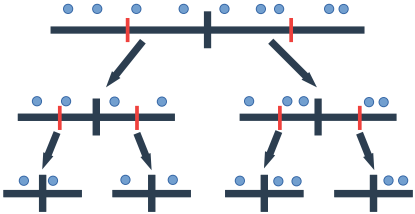

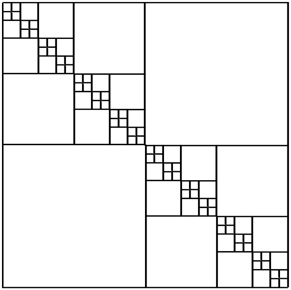

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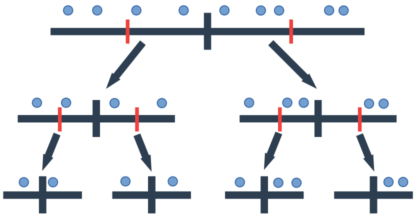

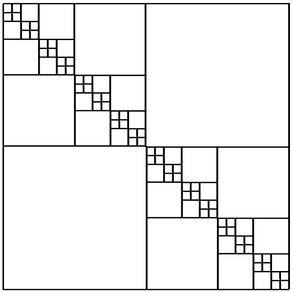

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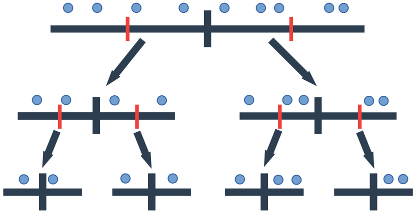
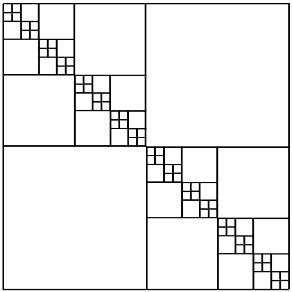
- Want k fixed (or *slowly growing*) as $N \to \infty$





Application: nonequilibrium Dyson equation from quantum many-body physics

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- Exponential complexity goes into interaction kernel (self-energy Σ) between G_{jk}, which is truncated at some order

Nonequilibrium Dyson equations

$$(-\partial_\tau - h(0))\, G^{\mathrm{M}}(\tau) - \int_0^\beta d\bar\tau\, \Sigma^{\mathrm{M}}(\tau-\bar\tau) G^{\mathrm{M}}(\bar\tau) = 0 \tag{1}$$

$$(-\mathrm{i}\partial_{t'} - h(t'))\, G^R(t,t') - \int_{t'}^t d\bar t\, G^R(t,\bar t)\Sigma^R(\bar t,t') = 0 \tag{2}$$

$$(\mathrm{i}\partial_t - h(t))\, G^{\rceil}(t,\tau) - \int_0^t d\bar t\, \Sigma^{\mathrm{R}}(t,\bar t) G^{\rceil}(\bar t,\tau) = \int_0^\beta d\bar\tau\, \Sigma^{\rceil}(t,\bar\tau) G^M(\bar\tau-\tau) \tag{3}$$

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$$G^{\mathrm{M}}(-\tau) = \xi G^{\mathrm{M}}(\beta-\tau) \tag{5}$$

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Focus on this ⟶

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$$y'(t) + p(t)y(t) + \int_0^t k(t,s)y(s)\, ds = 0$$

$$y(0) = y_0$$

$$t \in [0,T]$$

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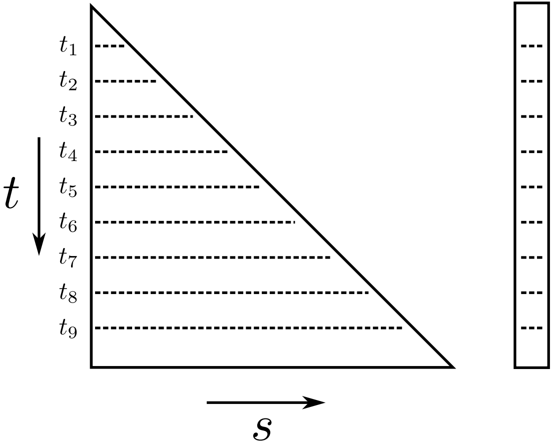

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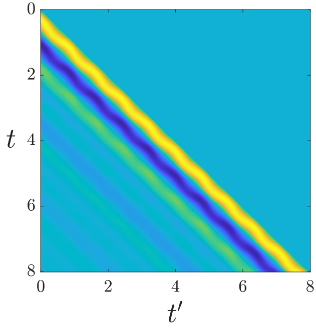

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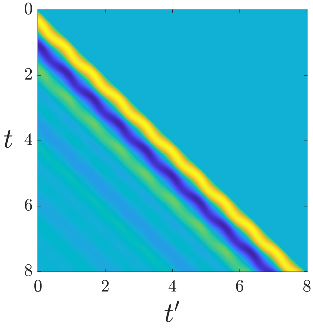

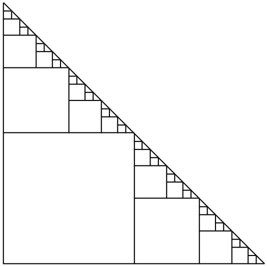

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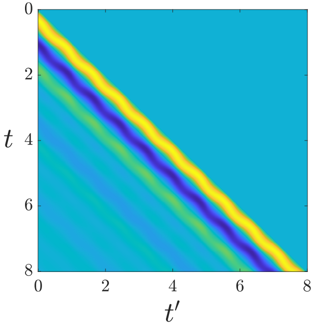

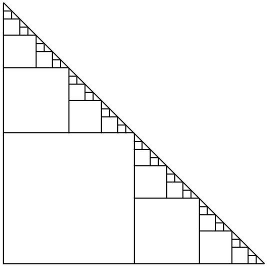

- Have efficient method of building HODLR representation of $\Sigma^R(t, t')$

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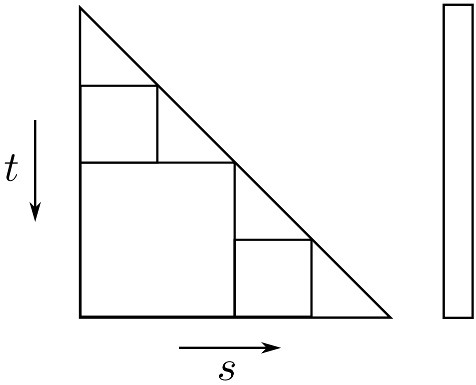

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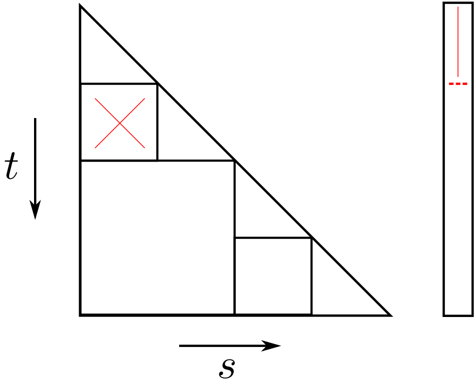

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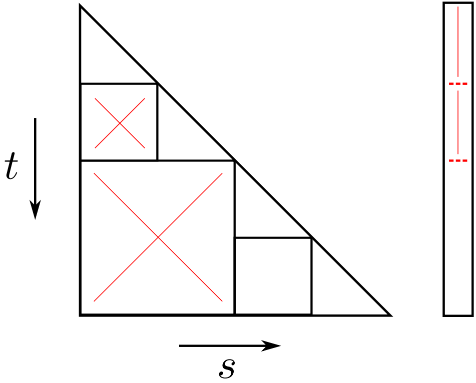

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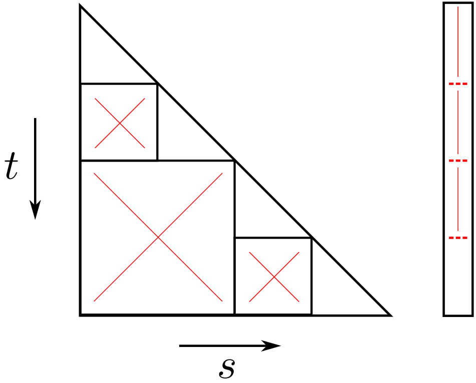

Hierarchical low rank compression for the nonequilibrium Dyson equation

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